On the regularity of the exercise boundary for American options

Adrien BLANCHET [a,b]

[a] *CEREMADE, Université Paris Dauphine, place de Lattre de Tassigny, 75775 Paris Cédex 16, France*
[b] *CERMICS, ENPC, 6–8 avenue Blaise Pascal, Champs-sur-Marne, 77455 Marne-la-Vallée Cédex 2, France.*

Abstract

We consider a complete market which rules out arbitrage. In the Black–Scholes model with local volatility the pricing of American option yields a parabolic obstacle problem. This paper is devoted to local regularity results of the exercise boundary for an American option on one underlying asset. We give an energy and a density criterion to characterise the subsets of the exercise boundary which are Hölder continuous with exponent $\frac{1}{2}$. As an illustration we apply these results to the generalised Black–Scholes model where the volatility and the interest rate do not depend on time. In this case we prove that the exercise boundary of the American put and call options are Hölder continuous with exponent $\frac{1}{2}$.

AMS Classification: 35R35.
Keywords: American option, parabolic obstacle problem, exercise region, exercise boundary, free boundary.

1. Introduction

We consider a complete market which rules out arbitrage (*i.e.* the market rules out the possibility to make an instantaneous risk-free benefit).

In this market we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(W_t)_{t\geq 0}$ the standard Brownian motion under the risk neutral probability $\mathbb{P}$, $(\mathcal{F}_t)_{t\geq 0}$ the $\mathbb{P}$-completion of the natural filtration of $(W_t)_{t\geq 0}$, $(S_t)_{t\geq 0}$ a one-dimensional price process and r the short rate of interest (to be quite general from a mathematical point of view, we assume here that r depends on the time t and also on the price S_t of the asset. Also see [6] for such kind of model). In the Black–Scholes model with local volatility (see [11,5]), we assume that S_t satisfies the following stochastic differential equation

$$dS_t = r(S_t, t)\, S_t\, dt + \sigma(S_t, t)\, S_t\, dW_t\,, \quad \forall t \in [0, T]\,. \tag{1.1}$$

Here σ is called *the local volatility*.

Email address: `blanchet@ceremade.dauphine.fr` (Adrien BLANCHET).
URL: `http ://www.ceremade.dauphine.fr/∼blanchet` (Adrien BLANCHET).

An American option is the right to sell (or buy) during a period of time a share of a specific common stock, called *the underlying asset*, at a prescribed price P^0. Here P^0 depends on the price of this asset S_t and the time t. Given a positive time T, called *the maturity*, the American option allows to make the following *pay-off* if one sells (respectively buys) an asset at any time $t \in [0,T]$:

$$\phi(S_t,t) := \max\{0, P^0(S_t,t) - S_t\}\,, \quad (\text{resp.} \quad \phi(S_t,t) := \max\{0, S_t - P^0(S_t,t)\})\,. \tag{1.2}$$

Let us denote by $(S_s^{x,t})_{s\in[t,T]}$ the process solution of (1.1) with initial condition $S_t^{x,t} = x$. For a value of the risky asset equal to x at time t the price of the American option at time t is given by the following optimal stopping time problem:

$$\Pi(x,t) := \sup_{\tau\in\Theta_{[t,T]}} \mathbb{E}\left[\exp\left(-\int_t^\tau r(S_s^{x,t},s)ds\right)\phi(S_\tau^{x,t},\tau)\right] \tag{1.3}$$

where $\Theta_{[t,T]}$ is the set of all $\mathcal{F}_t$-stopping time τ with value in $[t,T]$. The rigorous financial interpretation of the American option pricing as an optimal stopping problem has been proven by A. Bensoussan (see [3]) and I. Karatzas (see [18]). The first formulation of the pricing of American option in terms of optimal stopping problem is a prior work of H.P. Mc Kean in [22].

Let us define for $(x,t) \in \mathbb{R}\times[0,T]$

$$u(x,t) := \Pi(e^x, T-t) - \phi(e^x, T-t)$$

where Π is the price of the American option given by (1.3) and ϕ is the pay-off given by (1.2). In [4] (Théorème 4.1) A. Bensoussan and J.–L. Lions proved that the function u is the solution in $\mathbb{R}\times[0,T]$ in a variational sense of:

$$\begin{cases} Lu = f\,\mathbb{1}_{\{u>0\}}, \\ u \geq 0, \end{cases} \tag{1.4}$$

with initial condition $u(\cdot,0) = 0$. Here $\mathbb{1}_{\{u>0\}}$ denotes the characteristic function of the set $\{u > 0\} := \{(x,t)\in Q_R(P_0) : u(x,t) > 0\}$ and the parabolic operator L is defined by

$$Lu := a(\cdot,\cdot)\,\frac{\partial^2 u}{\partial x^2} + b(\cdot,\cdot)\,\frac{\partial u}{\partial x} + c(\cdot,\cdot)\,u - \frac{\partial u}{\partial t}\,,$$

$$\begin{aligned} &\text{with } a(x,t) := \frac{\sigma^2}{2}(e^x, T-t),\ b(x,t) := r(e^x,T-t) - \frac{\sigma^2}{2}(e^x,T-t),\ c(x,t) := -r(e^x,T-t) \\ &\text{and } -f(x,t) := (L\psi)(x,t) \text{ where } \psi(x,t) := \phi(e^x, T-t)\,. \end{aligned} \tag{1.5}$$

Their assumptions are stronger than Assumption (1.6) but they provide rigorously the link between American option pricing and parabolic obstacle problem. This work was taken further by P. Jaillet, D. Lamberton and B. Lapeyre, still in a variational interpretation but with stronger assumption than what we make here. More recently this link has been justified in the framework of viscosity solutions under Assumptions (1.6): by S. Villeneuve in [25] for the classical Black–Scholes model and by G. Rapuch in [23] for the Black–Scholes model with local volatility. The proof that the stochastic formulation (1.3) is actually a solution almost everywhere of (1.4) is still open. However the author believes that one could follow up the proof of Theorem 2.1 in [7] in the viscosity framework. For this purpose we need a Harnack inequality and Schauder interior estimates for viscosity solutions.

We are interested in local qualitative properties of the solution of the one-dimensional parabolic obstacle problem. Namely, for given $P_0 = (x_0,t_0) \in \mathbb{R}^2$ and $R > 0$ we look at the regularity in the open parabolic cylinder,

$$Q_R(P_0) := \{\,(x,t) \in \mathbb{R}^2 \,:\, |x-x_0| < R \text{ and } |t-t_0| < R^2\,\}\,,$$

of the solutions in

$$W^{2,1;1}_{x,t}(Q_R(P_0)) := \Big\{ u \in L^1(Q_R(P_0)) \,:\, \Big(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}\Big) \in (L^1(Q_R(P_0)))^3 \Big\}$$

of the one-dimensional parabolic obstacle problem (1.4)

Our main assumption is the following, concerning the uniform parabolicity (*i.e. the completeness of the market*) and non-degeneracy of the operator and also the regularity of the coefficients and the function f:

$$\begin{cases} \textit{there exists a constant } \delta_0 > 0 \textit{ such that for any } (x,t) \in Q_R(P_0)\,,\ a(x,t) \geq \delta_0 \ \textit{ and } \ f(x,t) \geq \delta_0\,, \\ a,\, b,\, c \textit{ and } f \textit{ belong to } \mathcal{H}^\alpha(Q_R(P_0)) \quad \text{for some } \alpha \in (0,1)\,, \end{cases} \tag{1.6}$$

where

$$\mathcal{H}^\alpha(Q_R(P_0)) := \Big\{ f \in \mathcal{C}^0 \cap L^\infty(Q_R(P_0)) \,:\, \sup_{\substack{(x,t),(y,s) \in Q_R(P_0) \\ (x,t) \neq (y,s)}} \frac{|f(x,t) - f(y,s)|}{(\,|x-y|^2 + |t-s|\,)^{\alpha/2}} < \infty \Big\}.$$

The regularity assumptions on the coefficients (*i.e.* on r and σ) are not too restrictive and they are usually admitted for local volatility models.

By [16], under Assumption (1.6), the equation (1.4) has a unique solution for suitable initial datum and boundary conditions. From standard regularity theory for parabolic equations (see [21,14,19]), it is known that u is continuous. The set $\{u = 0\}$ is then closed in $Q_R(P_0)$. In the theory of obstacle problem the sets $\{u = 0\}$ and $\Gamma := Q_R(P_0) \cap \partial\{u = 0\}$ are respectively called the *coincidence set* and the *free boundary* of the parabolic obstacle problem (1.4). In financial mathematics the sets $\{u = 0\} = \{\Pi = \phi\}$ and Γ its boundary are respectively called the *exercise region* and the *optimal exercise boundary.* The set $\{\Pi > \phi\}$ is called the *continuation region.*

We first need to define some local qualitative properties of a curve. Consider a curve in $\mathbb{R}^2$ defined by the equation $x = g(t)$ for some function g. For every time $t_1 < t_2$ we define the Hölder space

$$\mathcal{C}^{\frac{1}{2}}(t_1,t_2) := \Big\{ g \in \mathcal{C}^0(t_1,t_2) \,:\, \sup_{t \in (t_1,t_2)} |g(t)| + \sup_{\substack{t,s \in (t_1,t_2) \\ t \neq s}} \frac{|g(t) - g(s)|}{|t-s|^{\frac{1}{2}}} < \infty \Big\}.$$

We need the notion of $\mathcal{C}^{\frac{1}{2}}$-graph, $\mathcal{C}^{\frac{1}{2}}$-subgraph and $\mathcal{C}^{\frac{1}{2}}$-uppergraph: let $P_0 \in \mathbb{R}^2$ and $R > 0$. Consider a subset $A \subset Q_R(P_0)$ and $P_1 = (x_1,t_1) \in A$.

(i) We say that A is locally a $\mathcal{C}^{\frac{1}{2}}$*-graph* near P_1 if there exists $\rho > 0$ and $g \in \mathcal{C}^{\frac{1}{2}}(t_1 - \rho^2, t_1 + \rho^2)$ such that $Q_\rho(P_1) \subset Q_R(P_0)$ and $A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) \,:\, x = g(t)\}$.

(ii) We say that A is locally near P_1 a $\mathcal{C}^{\frac{1}{2}}$*-subgraph* (respectively a $\mathcal{C}^{\frac{1}{2}}$*-uppergraph*) if there exists $\rho > 0$ and $g \in \mathcal{C}^{\frac{1}{2}}(t_1 - \rho^2, t_1 + \rho^2)$ such that $Q_\rho(P_1) \subset Q_R(P_0)$ and $A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) \,:\, x \leq g(t)\}$ (resp. $A \cap Q_\rho(P_1) = \{(x,t) \in Q_\rho(P_1) \,:\, x \geq g(t)\}$).

Under Assumption (1.6) if we consider a solution of (1.4) we can give a density characterisation of all point $P_1 = (x_1,t_1) \in \Gamma$. This criterion is based on the density $\theta(P_1)$ of the coincidence set $\{u = 0\}$ at the point $P_1 \in \Gamma$:

$$\theta(P_1) := \liminf_{r \to 0} \frac{|\{u = 0\} \cap Q_r(P_1)|}{|Q_r(P_1)|}$$

and the lower density $\theta^-(P_1)$ of $\{u = 0\}$ at $P_1 \in \Gamma$, defined by

$$\theta^-(P_1) := \liminf_{r \to 0} \frac{|\{u=0\} \cap Q_r^-(P_1)|}{|Q_r^-(P_1)|},$$

with $Q_r^-(P_1) := \{ (x,t) \in \mathbb{R}^2 \, : \, |x - x_1| < r \text{ and } 0 < t_1 - t < r^2 \}$.

(i) If $\theta^-(P_1) \neq 0$ we say that P_1 is a *regular point.* We denote by $\mathcal{R}$ the set of regular points.

(ii) If $\theta^-(P_1) = 0$ we say that P_1 is a *singular point.* We denote by $\mathcal{S}$ the set of singular points. Furthermore we define the set $\mathcal{S}_0$ of the singular points such that $\theta(P_1) = 0$.

We recall in Proposition 2.2 an energy criteria to characterise these points of Γ.

It is proved (see Proposition 5.8 in [7]) that for almost every time there is no point of $\mathcal{S} \setminus \mathcal{S}_0$. Namely, the set $I := \{t \in [-R^2, R^2] \, : \, \exists\, x \in [-R, R], (x,t) \in \Gamma \setminus (\mathcal{R} \cup \mathcal{S}_0)\}$ has zero Lebesgue measure. The main result of this paper deals with the regularity of $\mathcal{S}_0$ and $\mathcal{R}$.

Theorem 1.1 (Regularity property of $\mathcal{R}$ and $\mathcal{S}_0$) *Under Assumption (1.6),*

(i) *the set of regular points, $\mathcal{R}$, is locally a $C^{\frac{1}{2}}$-graph. Furthermore around points of $\mathcal{R}$ the coincidence set is locally described by a $C^{\frac{1}{2}}$-subgraph or by a $C^{\frac{1}{2}}$-uppergraph,*

(ii) *the set $\mathcal{S}_0$ is locally contained in a $C^{\frac{1}{2}}$-graph.*

Remark 1.2 *This result is local and is not true up to the maturity. There is a large literature on the study of the regularity of Γ close to the maturity (see [20,2]). F. Charretour and R. Viswanathan were the first to notice that the exercise boundary cannot be $\mathcal{C}^{\frac{1}{2}}$ up to the maturity.*

This question of the regularity of the free boundary is crucial in financial mathematics and in particular in the numerical computation of Γ. As an illustration in [12], N. El Karoui proved in the framework of optimal theory that

$$\Pi(x,t) = \mathbb{E}\left[\exp\left(-\int_t^{\tau^*} r(S_s^{x,t}, s)ds\right) \phi(S_{\tau^*}^{x,t}, \tau^*)\right] \quad \text{where} \quad \tau^* := \{\inf \tau \geq t \, : \, \Pi(S_\tau, \tau) = \phi(S_\tau, \tau)\}\,.$$

So the knowledge of the exercise boundary gives the best strategy for the owner of the option. Unfortunately we cannot give an explicit formula to describe the free boundary, even in the constant coefficients case. However, a better understanding of the behaviour of the free boundary is a crucial question in financial markets. This problem also appear in [1], where Y. Achdou solves a calibration problem. It is an inverse problem where he evaluates the volatility σ by the knowledge of the price of American options on the financial market. He needs regularity of the exercise boundary to control his numerical computations.

P. Van Moerbeke studied in [24], call options in the classical Black–Scholes model. He subsequently proved that the exercise boundary has a continuous time derivative except at the maturity. In [15], A. Friedman considered the case where the variable coefficients depend on time and space and are continuously differentiable. He proved under assumptions on the change of sign of the initial data that the exercise boundary consists of a finite number of curves $t \to s_i(t)$ piecewise monotone and continuous. He also proved that $\frac{ds_i(t)}{dt}$ exists and is continuous in every t-interval where s_i is strictly monotone. The proof of these results need to differentiate the equation with respect to x and to apply the maximum principle to u_x. This proof cannot apply to parabolic operator with Hölder coefficients.

Recently in [8], L. Caffarelli, A. Petrosyan and H. Shahgholian considered the case of the parabolic potential problem (*i.e.* with constant coefficients in any dimension and without any sign assumptions on the solution). They give an energy and a density criterion to classify the points of the free boundary in two sets: the *regular points* and the *singular points.* Their density criterion is similar to the density characterisation given above and the energy criterion is similar to the one given in Proposition 2.2. They prove that around regular points the free boundary in $\mathcal{C}^\infty$. Their proof apply to the variable coefficients case if the coefficients are Lipchitz (in this case the regular set is Lipschitz. But this has not been done) and they do not study the set of singular points. However their method has been extended in [7] by J. Dolbeault, R. Monneau and the author in the variable coefficients case with Hölder continuity. They give an energy and a density criterion to characterise the region of the free boundary such that the time derivative of the solution is continuous. They study the set of singular points in order to prove that the time derivative of the solution is continuous almost everywhere. For the convenience of the reader all the results and notation of [7] we will use in the proofs are recalled in Section 2. Section 3 is consecrated to the proof of the main theorem. We study the regular case in Section 3.1 and the singular case in Section 3.2. In Section 4 we illustrate our result by classifying the possible shapes of the continuation region and the regularity of the optimal exercise boundary in the Black-Scholes model with homogeneous diffusion (case where σ and r do not depend on time) with generic pay-off functions. In Section 5 we apply our results to the American put and call options in this model.

Notation. We will use u_t, u_x and u_{xx} respectively for $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$. For any domain $D \subset \mathbb{R}^2$, we define $W^{2,1;q}_{x,t}(D) := \left\{ u \in L^q(D) \, : (u_x,\, u_{xx},\, u_t) \in (L^q(D))^3 \right\}$. And we will write $u \in W^{2,1;q}_{x,t;\mathrm{loc}}(D)$ if $u \in W^{2,1;q}_{x,t}(K)$ for all compact $K \subset\subset D$. The heat operator will be abbreviated to H, $Hu := u_{xx} - u_t$.

2. Known results

Under Assumption (1.6) the solutions of (1.4) are bounded in $W^{2,1;\infty}_{x,t}(Q_{R'}(P_0))$ for all $R' < R$ (see Theorem 2.1 in [7]). The theory we develope here and in [7] lies on the founding idea of L. Caffarelli (see [9]). The idea is to use blow-up sequences, which are kinds of zooms, and to look at the "infinite zoom". Namely consider $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ a sequence of point of Γ, $r > 0$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence converging to 0. The *blow-up sequence* $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ associated to a function $u : Q_r(0) \longrightarrow \mathbb{R}$ is the sequence defined by

$$u^{\varepsilon_n}_{P_n}(x,t) = \varepsilon_n^{-2}\, u\left(x_n + \varepsilon_n x\sqrt{\frac{a(P_n)}{f(P_n)}}, t_n + \varepsilon_n^2 t\,\frac{1}{f(P_n)}\right) \quad \forall n \in \mathbb{N}, \quad \forall (x,t) \in Q_{r/\varepsilon_n}(0)\,.$$

Notice that the parabolic scaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$ transforms the parabolic cylinder $Q_\lambda(0)$ into the parabolic cylinder $Q_1(0)$.

Due to $W^{2,1;\infty}_{x,t}$ *a priori* regularity estimates, up to the extraction of a subsequence, blow-up sequences uniformly converge on every compact to a function $u^0 \in W^{2,1;\infty}_{x,t;\mathrm{loc}}(\mathbb{R}^2)$. We recall Lemma 2.6 in [7] (see also Lemma 5.1 in [8]):

Lemma 2.1 (Non-degeneracy lemma) *Under Assumption (1.6), consider a solution u of (1.4) in $Q_R(P_0)$. Let $R' \in (0,R)$, $P_1 \in \overline{\{u > 0\}}$ be such that $Q^-_r(P_1) \subset Q_{R'}(P_0)$ for some $r > 0$ small enough. There exist two positive constants $\bar{C}$ and $\bar{r} > 0$ such that if $Q_{\bar{r}}(P_1) \cap \{u = 0\} \neq \emptyset$:*

$$r \le \bar{r} \quad \Longrightarrow \quad \sup_{Q^-_r(P_1)} u \ge \bar{C}\, r^2\,.$$

The constants $\bar{C}$ and $\bar{r}$ only depend on R' and the parabolic operator L.

The non-degeneracy lemma was first proved by L. Caffarelli in [9] for the elliptic obstacle problem. Its proof lies on the maximum principle.

More precisely (see proof of Proposition 3.2, Proposition 3.3 and Proposition 2.9 in [7]) u^0 is solution in $\mathbb{R}^2$ of the following global parabolic obstacle problem:

$$
\begin{cases} \dfrac{\partial^2 u^0}{\partial x^2}(x,t) - \dfrac{\partial u^0}{\partial t}(x,t) = \mathbb{1}_{\{u^0>0\}}(x,t) & \text{a.e. } (x,t)\in\mathbb{R}^2 \\ u^0(x,t)\geq 0 \end{cases}
$$

and $0\in\partial\{u^0>0\}$. Furthermore

$$
\lim_{n\to\infty}\mathbb{1}_{\{u^{\varepsilon_n}_{P_n}=0\}} = \mathbb{1}_{\{u^0=0\}} \quad \text{a.e. in } \mathbb{R}^2\,. \tag{2.1}
$$

Moreover if we consider a blow-up sequence in a fixed point P_1 (*i.e.* $P_n\equiv P_1$), the blow-up limit is one of the following functions

$$
\begin{aligned} u^0_+(x,t) &:= \frac{1}{2}\,(\max\{0,x\})^2, \\ u^0_-(x,t) &:= \frac{1}{2}\,(\max\{0,-x\})^2 \end{aligned} \quad \text{and} \quad u^0_m(x,t) := \begin{cases} m\,t + \dfrac{1+m}{2}\,x^2 & \text{if}\, t<0\,, \\ \max\left\{0, t\,U_m\left(\dfrac{|x|}{\sqrt{t}}\right)\right\} & \text{if}\, t\geq 0\,, \end{cases} \tag{2.2}
$$

where $m\in[-1,0]$ and U_m is explicitly given in [7] (see Theorem 3.9 in [7] see also Lemma 6.3 in [8]).

The crucial difficulty in this characterisation of blow-up limits in fixed points is to prove their scale-invariance. For this purpose, in [26], G. Weiss introduced a monotonicity formula for the elliptic obstacle problem. For the parabolic obstacle problem we define the energy as follows:

Let $Q_r(P_1)\subset Q_R(P_0)\subset\mathbb{R}^2$. With $P_1=(x_1,t_1)$, and a, f the functions involved respectively in the definition of the operator L and in Equation (1.4). Consider a nonnegative cut-off function $\psi\in\mathcal{C}^\infty(\mathbb{R})$ such that $\psi\equiv 1$ on $\left(-\frac{1}{2}\sqrt{\frac{f(P_1)}{a(P_1)}},\frac{1}{2}\sqrt{\frac{f(P_1)}{a(P_1)}}\right)$ and $\psi\equiv 0$ on $\left(-\infty,\sqrt{\frac{f(P_1)}{a(P_1)}}\right]\cup\left[\sqrt{\frac{f(P_1)}{a(P_1)}},\infty\right)$ and define $\psi_r(x):=\psi(r\,x)$ and the function v (which depends on u, P_1 and r) for all $(x,t)\in\mathbb{R}\times(-r^2\,f(P_1),r^2\,f(P_1))$ by

$$
v(x,t) := u\left(x_1 + x\sqrt{\tfrac{a(P_1)}{f(P_1)}}, t_1 + \frac{t}{f(P_1)}\right)\cdot\psi_r(x) \;\text{ if }\; |x|\leq r\sqrt{\tfrac{f(P_1)}{a(P_1)}}\,, \quad v\equiv 0 \;\text{ otherwise}. \tag{2.3}
$$

For all $t\in(-r^2\,f(P_1),0)$, define

$$
\mathcal{E}_{u,P_1}(\tau,r) := \int_{\mathbb{R}}\left\{\left[\frac{1}{-\tau}\left(\left|\frac{\partial v}{\partial x}\right|^2 + 2\,v\right) - \frac{v^2}{\tau^2}\right]G\right\}(x,\tau)\,dx - \int_\tau^0\frac{1}{s^2}\int_{\mathbb{R}}\{(Hv-1)\,(\mathcal{L}v)\,G\}\,(x,s)\,dx\,ds\,,
$$

with $Hv:=v_{xx}-v_t$, $\mathcal{L}v := -2\,v + x\cdot v_x + 2\,t\,v_t$ and $G(x,t):=(2\pi(-t))^{-\frac{1}{2}}\exp\left(-x^2/(-4t)\right)$. For this energy we have (Proposition 3.4, Lemma 3.7 and Proposition 4.1 in [7]):

Proposition 2.2 (Monotonicity formula) *Let $Q_r(P_1)\subset Q_R(P_0)$. Under Assumption (1.6), if u is a solution of (1.4) and v defined in (2.3), then for a given $r>0$ the function $\tau\mapsto\mathcal{E}_{u,P_1}(\tau,r)$ is a non-increasing function, which is bounded from below and bounded in $W^{1,\infty}(-1,0)$. Furthermore for $r>0$ and a given $\tau_0<0$, $P\mapsto\mathcal{E}_{u,P}(\tau_0,r)$ is continuous. And if $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ is a blow-up sequence associated to u in a fixed point P_1 and $u^0_{P_1}$ a blow-up limit of $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ then*

$$
\forall r>0 \quad \mathcal{E}_{u,P_1}(\varepsilon_n^2\tau,r) = \mathcal{E}_{u^{\varepsilon_n}_{P_1},0}(\tau,\varepsilon_n r)\to\mathcal{E}_{u^0,0}(\tau,0)\in\{\frac{\sqrt{2}}{2},\sqrt{2}\}\,. \tag{2.4}
$$

Moreover if, for $r>0$, $\lim_{\tau\to 0}\mathcal{E}_{u,P_1}(\tau,r)=\frac{\sqrt{2}}{2}$, then P_1 is regular. And if, for $r>0$, $\lim_{\tau\to 0}\mathcal{E}_{u,P_1}(\tau,r)=\sqrt{2}$, then P_1 is singular.

As a consequence : $\mathcal{S}$ is a closed set, and $\mathcal{R}=\Gamma\setminus\mathcal{S}$ is open in Γ (Lemma 4.2 in [7]).

Remark 2.3 *This energy characterisation of the sets $\mathcal{R}$ and $\mathcal{S}$ gives a criterion to apply Theorem 1.1 (i). It is sufficient to prove that $\lim_{\tau\to 0}\mathcal{E}_{u,P_1}(\tau,r)<\sqrt{2}$. It can be interesting for practical financial applications. The derivative with respect to the initial condition, u_x, is known as the Delta in Greeks formulae and can be computed numerically with Monte-Carlo methods. See also [13] for a recent approach of the calculus of Delta using Malliavin calculus.*

For a further inspection of the singular set J. Dolbeault, R. Monneau and the author introduce in [7] a monotonicity formula for singular points for the parabolic obstacle problem. Consequently they prove the uniqueness of blow-up limit in singular points. Namely, that under Assumption (1.6), if u is a solution of (1.4) and $(P_n)_{n\in\mathbb{N}}$ a sequence of singular points, then there exists a unique $m \in [-1,0]$ such that for any sequence, $(\varepsilon_n)_{n\in\mathbb{N}}$, converging to 0, the whole blow-up sequence, $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$, locally uniformly converges to u^0_m, where u^0_m is defined in (2.2) (see Proposition 5.5 and Lemmata 5.6 and 5.7 in [7]).

The main theorem of [7] (see Corollaries 6.4 and 6.7 in [7]) is the following:

Theorem 2.4 (Continuity of the time derivative) *Under Assumption (1.6) consider a solution u of (1.4). If $P_1 \in \mathcal{R} \cup \mathcal{S}_0$ then*

$$\lim_{P \to P_1} \frac{\partial u}{\partial t}(P) = 0\,.$$

3. Proof of the main theorems

In Section 3.1 we study the regularity of the regular set. We first prove the uniqueness of blow-up limit in regular points. Then we prove a uniform holderian born on regular points. These two results lead to our statement. In Section 3.2, as the uniqueness of blow-up limit in singular points is a result of [7] (see above), we just have to prove a uniform holderian bound on singular points to conclude.

3.1. *Proof of Theorem 1.1 (i)*

Lemma 3.1 (Uniqueness of blow-up limits in regular points) *Under Assumption (1.6), consider a solution of (1.4). If $P_1 \in \mathcal{R}$ then there exists a unique $\gamma \in \{+,-\}$ such that for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, the whole blow-up sequence $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ in the fixed point P_1 locally uniformly converges to u^0_γ, where u^0_+ and u^0_- are defined in (2.2).*

Proof. By the energetic characterisation of $\mathcal{R}$ (Proposition 2.2), up to the extraction of a subsequence $(u^{\varepsilon_n}_{P_1})_{n\in\mathbb{N}}$ converges to u^0_+ or u^0_-. Assume by contradiction that there are two subsequences $(\varepsilon_{n'})_{n'\in\mathbb{N}}$ and $(\varepsilon_{n''})_{n''\in\mathbb{N}}$ such that $(u^{\varepsilon_{n'}}_{P_1})_{n'\in\mathbb{N}}$ converges to u^0_+ and $(u^{\varepsilon_{n''}}_{P_1})_{n''\in\mathbb{N}}$ converges to u^0_-. We have $u^0_+(1,0) = \frac{1}{2}$ and $u^0_-(1,0) = 0$. By continuity of $\varepsilon \mapsto u^\varepsilon_{P_1}$, this implies that there exists another subsequence $(\tilde\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} u^{\tilde\varepsilon_n}_{P_1}(1,0) = \frac{1}{4}$. But this property is satisfied neither by u^0_+ nor by u^0_-, however $P_1 \in \mathcal{R}$. We obtain thus a contradiction. □

Definition 3.2 ($\mathcal{R}_+$ and $\mathcal{R}_-$) *Let $P_1 \in \mathcal{R}$. The set of points such that the blow-up limit in P_1 is u^0_+ is denoted $\mathcal{R}_+$. The set of points such that the blow-up limit in P_1 is u^0_- is denoted $\mathcal{R}_-$.*

For any $\delta > 0$, if not empty, let us define the closed set:

$$\mathcal{R}_\delta = \{P \in \mathcal{R}, \quad \mathrm{dist}(P, \mathcal{S}) \geq \delta\}\,.$$

Lemma 3.3 *For any $\delta > 0$, $\mathcal{R}_\delta$ is locally contained in a $C^{\frac{1}{2}}$-graph. More precisely, for any $\delta > 0$, there exists a constant $M(\delta) > 0$ such that,*

$$\sup_{(x,t)\in\mathcal{R}_\delta} \sup_{\substack{(x',t')\in\mathcal{R}_\delta \\ (x',t')\neq(x,t)}} \frac{|x'-x|}{\sqrt{|t'-t|}} \leq M(\delta)\,.$$

Proof. Assume by contradiction that there are two sequences of points $(P_n = (x_n, t_n))_{n\in\mathbb{N}}$ and $(P'_n = (x'_n, t'_n))_{n\in\mathbb{N}}$ in $\mathcal{R}_\delta$ converging to a point $P_\infty \in \mathcal{R}_\delta$, such that

$$\lim_{n\to\infty} \frac{|x'_n - x_n|}{\sqrt{|t'_n - t_n|}} = +\infty\,.$$

The blow-up sequence $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ with $\varepsilon_n := \sqrt{(x'_n - x_n)^2 + |t'_n - t_n|}$ converges, up to the extraction of a subsequence, to a function u^0. Define $r_n > 0$ such that $Q_{r_n}(x_n + \varepsilon_n x, t_n + \varepsilon_n^2 t) \subset \{u > 0\}$, we have:

$$\frac{d}{dt} u^{\varepsilon_n}_{P_n}(x,t) = \frac{du}{dt}(x_n + \varepsilon_n x, t_n + \varepsilon_n^2 t)\,.$$

By Schauder interior estimates u_t is bounded in $\mathcal{H}^\alpha$, and the corresponding bound is uniform under scaling (also see Theorem 2.1 in [7]) so we can pass to the limit in u_t. The right term converges to 0 because of the continuity of the time derivative (Theorem 2.4) applied to $(x_n + \varepsilon_n x, t_n + \varepsilon_n^2 t)$ which tends to $P_\infty \in \mathcal{R}$. Therefore $u^0_t \equiv 0$.

The sequence of generic term

$$\nu_n := \left(\frac{x_n - x'_n}{\varepsilon_n}, \frac{t_n - t'_n}{\varepsilon_n^2} \right) \in \partial Q_1(0) \cap \partial\{u^{\varepsilon_n}_{P_n} = 0\}$$

converges to a point $\nu = (x_\nu, t_\nu) \in \partial Q_1(0)$. By non-degeneracy lemma (Lemma 2.1) there exists a positive constant $\bar{C}$ such that for all $r > 0$ small enough

$$\bar{C}\, r^2 \le \sup_{Q_r(\nu_n)} u^{\varepsilon_n}_{P_n} \to \sup_{Q_r(\nu)} u^0\,.$$

So ν belongs to $\partial\{u^0 = 0\}$.

To summarise $0 \in \partial\{u^0 = 0\}$ by non-degeneracy lemma (Lemma 2.1), $\nu \in \partial\{u^0 = 0\}$ by the above demonstration, $u^0_x \equiv 0$ on Γ because u is non-negative and $u^0_t \equiv 0$ by Theorem 2.4. This implies that

$$\begin{aligned} u^0(x,t) &= \frac{1}{2}\left(\max\{0,-x\}\right)^2 + \frac{1}{2}\left(\max\{0, x - x_\nu\}\right)^2 \quad \text{if} \quad x_\nu > 0\,, \\ u^0(x,t) &= \frac{1}{2}\left(\max\{0,x\}\right)^2 + \frac{1}{2}\left(\max\{0, x + x_\nu\}\right)^2 \quad \text{if} \quad x_\nu < 0\,. \end{aligned} \tag{3.1}$$

We will prove with an energy argument that this cannot be true.

The point P_∞ is in $\mathcal{R}_\delta$. So by the energy characterisation of the regular points (Proposition 2.2)

$$\lim_{\substack{\tau\to 0\\ \tau<0}} \mathcal{E}_{u,P_\infty}(\tau, r) = \frac{\sqrt{2}}{2} \quad \forall r > 0\,.$$

Hence, for a given $r > 0$, as $\tau \mapsto \mathcal{E}_{u,P}(\tau, r)$ is continuous (Proposition 2.2) for any $\delta > 0$, we can find $\tau_0 < 0$ such that

$$\mathcal{E}_{u,P_\infty}(\tau, r) \le \frac{\sqrt{2}}{2} + \frac{\delta}{4} \quad \forall \tau \in (\tau_0, 0),\, \forall r > 0\,. \tag{3.2}$$

But by (3.1) we know explicitly u^0 and we compute directly for all $\tau < 0$

$$\mathcal{E}_{u^0,0}(\tau, 0) = \frac{\sqrt{2}}{2} + \eta(\tau)$$

where $\eta(\tau)$ is positive and only depend on τ. By scale-invariance of $\mathcal{E}_{u,P}$ (Proposition 2.2)

$$\lim_{n\to\infty} \mathcal{E}_{u,P_n}(\varepsilon_n^2 \tau, r) = \lim_{n\to\infty} \mathcal{E}_{u^{\varepsilon_n}_{P_n},0}(\tau, \varepsilon_n r) = \mathcal{E}_{u^0,0}(\tau, 0) = \frac{\sqrt{2}}{2} + \eta(\tau)\,.$$

So for any $\delta > 0$ and τ_0 given in (3.2) there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\forall r > 0, \quad \mathcal{E}_{u,P_n}(\varepsilon_n^2 \tau, r) \geq \frac{\sqrt{2}}{2} + \eta(\tau) - \frac{\delta}{2} \quad \text{and} \quad \tau_0 < \varepsilon_n^2 \tau < 0\,. \tag{3.3}$$

However, $P \mapsto \mathcal{E}_{u,P}(\tau_0, r)$ is continuous for given τ_0 and $r > 0$ (Proposition 2.2). So for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies

$$\mathcal{E}_{u,P_n}(\tau_0, r) \leq \mathcal{E}_{u,P_\infty}(\tau_0, r) + \frac{\delta}{4} \quad \forall r > 0\,. \tag{3.4}$$

Combining (3.3) and (3.4) and because $\tau \mapsto \mathcal{E}_{u,P_n}(\tau, r)$ is non-increasing (Proposition 2.2) we have

$$\frac{\sqrt{2}}{2} + \eta(\tau) - \frac{\delta}{2} \leq \mathcal{E}_{u,P_n}(\varepsilon_n^2 \tau, r) \leq \mathcal{E}_{u,P_n}(\tau_0, r) \leq \mathcal{E}_{u,P_\infty}(\tau_0, r) + \frac{\delta}{4} \leq \frac{\sqrt{2}}{2} + \frac{\delta}{2} \quad \forall r > 0\,.$$

Which is a contradiction if we choose $\delta < \eta(\tau)$. □

We now prove

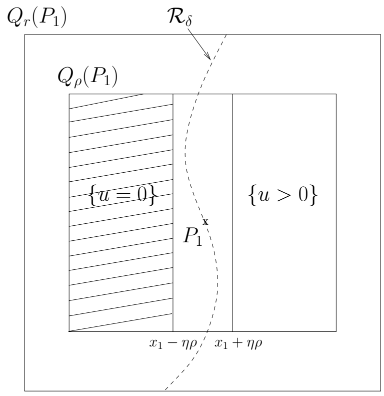
Lemma 3.4 *$\mathcal{R}_+$ and $\mathcal{R}_-$ are open subsets of Γ.*

We first need

Lemma 3.5 *If $P_1 = (x_1, t_1)$ belongs to $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) then for any $\eta \in (0,1)$, there exists ρ such that $u \equiv 0$ (resp. $u > 0$) in $(x_1 - \rho, x_1 - \eta\rho) \times (t_1 - \rho^2, t_1 + \rho^2)$ and $u > 0$ (resp. $u \equiv 0$) in $(x_1 + \eta\rho, x_1 + \rho) \times (t_1 - \rho^2, t_1 + \rho^2)$.*

Proof. By symmetry, we can assume that $P_1 = (x_1, t_1) \in \mathcal{R}_+$. The blow-up sequence in the fixed point P_1, $(u_{P_1}^{\varepsilon_n})_{n\in\mathbb{N}}$, converges to $\frac{1}{2}\left(\max\{0, x\}\right)^2$. The result is achieved thanks to (2.1) which states that $\mathbb{1}\{u_{P_1}^{\varepsilon_n} > 0\}$ converges to $\mathbb{1}\{u_{P_1}^0 > 0\}$. □

Proof of Lemma 3.4. By symmetry, we can assume that $P_1 = (x_1, t_1) \in \mathcal{R}_+$. As $\mathcal{R}$ is open, there exists $\delta > 0$ and $r > 0$ such that P_1 and $\Gamma \cap Q_r(P_1)$ are contained in $\mathcal{R}_\delta$. By Lemma 3.5, for any $\eta \in (0,1)$, there exists ρ such that $u \equiv 0$ in $(x_1 - \rho, x_1 - \eta\rho) \times (t_1 - \rho^2, t_1 + \rho^2)$ and u is positive in $(x_1 + \eta\rho, x_1 + \rho) \times (t_1 - \rho^2, t_1 + \rho^2)$.



Generic drawing for the proof of Theorem 1.1

By Lemma 3.3, $\Gamma \cap Q_\rho(P_1)$ is contained in a $C^{\frac{1}{2}}$-graph. This implies that for any $t \in (t_1 - \rho^2, t_1 + \rho^2)$, there exists a point $P = (g(t), t) \in \mathcal{R}$ such that $u(x,t) = 0$ if $x < g(t)$ and $u(x,t) > 0$ if $x > g(t)$ in $Q_\rho(P_1)$. Finally let $(u_P^{\varepsilon_n})_{n\in\mathbb{N}}$ be a blow-up sequence in the fixed point P with $\varepsilon_n \in \{(x,t) : x = g(t)\}$.

The blow-up limit is necessarily $\frac{1}{2}(\max\{0,x\})^2$. So $P \in \mathcal{R}_+$. So for any $t \in (t_1-\rho^2, t_1+\rho^2)$ all the points are in $\mathcal{R}_+$. $\square$

Theorem 1.1 (i) is a direct consequence of Lemma 3.3 and the topological property of $\mathcal{R}$. More precisely, locally around a point of $\mathcal{R}^+$ all the points are in $\mathcal{R}^+$ and the free boundary is locally $\mathcal{C}^{\frac{1}{2}}$. So around points of $\mathcal{R}^+$ the free boundary is locally a $\mathcal{C}^{\frac{1}{2}}$-subgraph. Respectively around points of $\mathcal{R}^-$, the free boundary is locally a $\mathcal{C}^{\frac{1}{2}}$-uppergraph.

3.2. *Proof of Theorem 1.1 (ii)*

Assume by contradiction that there are two sequences of points $(P_n = (x_n,t_n))_{n\in\mathbb{N}}$ and $(P'_n = (x'_n,t'_n))_{n\in\mathbb{N}}$ in $\mathcal{S}_0$ converging to a point P_∞ such that

$$\lim_{n\to\infty} \frac{|x'_n - x_n|}{\sqrt{|t'_n - t_n|}} = +\infty\,. \tag{3.5}$$

Recall that a consequence of the monotonicity formula (Theorem 2.2), $\mathcal{S}_0$ is closed. So P_∞ belongs to $\mathcal{S}_0$. We define $\varepsilon_n := \sqrt{(x'_n-x_n)^2 + |t'_n - t_n|}$. By uniqueness of the limit of the blow-up sequence in singular points, $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ if $(P_n)_{n\in\mathbb{N}} \in \mathcal{S}_0^{\mathbb{N}}$ (see Section 2), the blow-up sequence $(u^{\varepsilon_n}_{P_n})_{n\in\mathbb{N}}$ converges to $u^0(x,t) = \frac{1}{2}x^2$. The sequence of generic term

$$\nu_n := \left(\frac{x'_n - x_n}{\varepsilon_n}, \frac{t'_n - t_n}{\varepsilon_n^2}\right) \partial\{u^{\varepsilon_n}_{P_n} = 0\} \cap \partial Q_1(0)$$

converges to $\nu = (x_\nu, t_\nu)$. Because of (3.5), $\frac{|x_\nu|}{\sqrt{|t_n u|}} = \infty$ so ν belongs to $\{\{-1,0\} \cup 1, 0\}$. By non-degeneracy lemma (Lemma 2.1), 0 and ν belong to $\partial\{u^0 = 0\}$, which is a contradiction with $u^0(x,t) = \frac{1}{2}x^2$.

4. Applications to homogeneous diffusion

Let us apply our results to the Black–Scholes model with homogeneous diffusion (case where σ and r do not depend on time). In this case the link between the stochastic formulation (1.3) and the obstacle problem has been proved in [1] (Theorem 2.2).

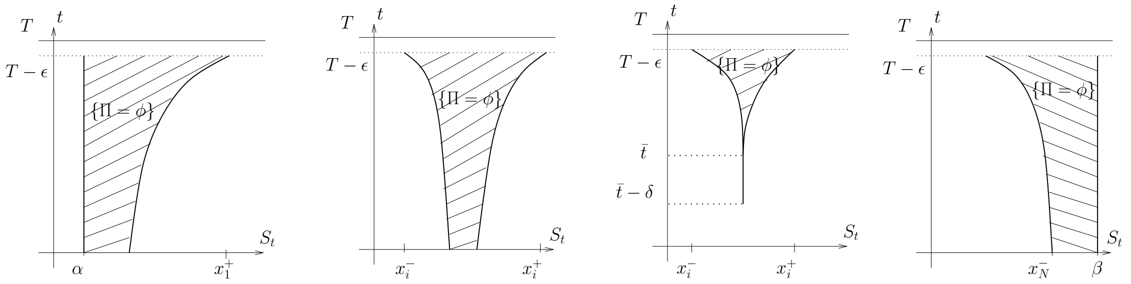
Theorem 4.1 (Exercise boundary in the Black-Scholes model) *In the generalised Black–Scholes model where σ and r do not depend on time, consider the price Π of the American option given by (1.3) with a finite maturity $T > 0$. Assume $\sigma \in L^\infty(\mathbb{R}\times[0,T])$ is such that $x\frac{d\sigma}{dx}$ is bounded in $L^\infty(\mathbb{R}\times[0,T])$ and the pay-off ϕ depends only on S_t, satisfies $|\phi(x)| \le M\, e^{M|x|}$, where $M > 0$. If*

$$\frac{\sigma^2}{2}x^2\frac{\partial^2\phi}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)x\frac{\partial\phi}{\partial x} + r\phi < 0$$

then for all $\varepsilon > 0$ and $[\alpha,\beta] \subset \mathbb{R}$ there exists $N \in \mathbb{N}$, $\tilde{t} \in [0, T-\varepsilon]$, $2N$ reals $\{x_i^- \le x_i^+\}_{i\in\{1,\dots,N\}}$ in $[\alpha,\beta]$ and $2N$ graphs of class $\mathcal{C}^{\frac{1}{2}}([0,\tilde{t}\,])$, $(g_i^- \le g_i^+)_{i\in\{1,\dots,N\}}$, such that for every $i \in \{1,\dots,N\}$

$$\{\Pi = \phi\} \cap ([x_i^-, x_i^+]\times[0,\tilde{t}\,]) = \{(x,t)\,:\, g_i^-(t) \le x \le g_i^+(t)\}\,.$$

Moreover if for a given $i \in \{1,\dots,N\}$ there exists $\bar{t} \in [0, T-\varepsilon]$ such that $g_i^-(\bar{t}\,) = g_i^+(\bar{t}\,)$ then there exists $\delta \in [0, T-\varepsilon-\bar{t}\,]$ such that $\{\Pi = \phi\} \cap ([x_i^-, x_i^+]\times[0,\bar{t}\,])$ is the straight line $g_i^-(\bar{t}\,)\times[\bar{t}-\delta, \bar{t}\,]$.



We first prove a slightly different lemma which bring us back to the framework of the Section 3.

Lemma 4.2 *Under Assumption (1.6), consider a solution of (1.4). If u is non-decreasing then for all $\varepsilon > 0$ and $[\alpha,\beta] \subset \mathbb{R}$ there exists $N \in \mathbb{N}$, $\tilde t \in [\varepsilon, T]$, $2N$ reals $\{x_i^- \le x_i^+\}_{i\in\{1,\dots,N\}}$ in $[\alpha,\beta]$ and $2N$ graphs of class $\mathcal{C}^{\frac12}([\varepsilon,\tilde t])$, $\{g_i^-, g_i^+\}_{i\in\{1,\dots,N\}}$, such that for every $i \in \{1,\dots,N\}$*

$$\{u=0\} \cap ([x_i^-, x_i^+] \times [\varepsilon, \tilde t]) = \{(x,t) \,:\, g_i^-(t) \le x \le g_i^+(t)\}\ .$$

Moreover if there exists $\bar t \in [\varepsilon, T]$ such that $g_i^-(\bar t) = g_i^+(\bar t)$ then there exists $\delta \in [0, T-\bar t]$ such that $\{u=0\} \cap ([x_i^-, x_i^+] \times [\bar t, T])$ is the straight line $g_i^-(\bar t) \times [\bar t, \bar t+\delta]$.

We first precise some properties of the exercise boundary around points of $\mathcal{S}_0$ and $\mathcal{R}$.

Lemma 4.3 *Under Assumption (1.6), consider a solution of (1.4).*

(i) *If $P_1 = (x_1,t_1) \in \mathcal{R}^+$ (resp. $\mathcal{R}^-$) then there exists $\rho > 0$ such that $u > 0$ in $(x_1-\rho, x_1) \times \{t=t_1\}$ (resp. $(x_1, x_1+\rho) \times \{t=t_1\}$) and $u \equiv 0$ in $(x_1, x_1+\rho) \times \{t=t_1\}$ (resp. $(x_1-\rho, x_1) \times \{t=t_1\}$).*

(ii) *If $P_1 = (x_1,t_1) \in \mathcal{S}_0$ then there exists $\rho > 0$ such that $u > 0$ in $((x_1-\rho, x_1) \cup (x_1, x_1+\rho)) \times \{t=t_1\}$.*

Furthermore for any time $t \in (0,T)$ there is a finite number of points of $(\mathcal{R} \cup \mathcal{S}_0) \times \{t\}$.

Proof. (i) By symmetry we can assume $P_1 \in \mathcal{R}^+$. Assume by contradiction that there exists $(P_n = (x_n, t_1))_{n\in\mathbb{N}}$ converging to $P_1 \in \mathcal{R}^+$ such that $u(P_n) = 0$. The blow-up sequence $(u_{P_1}^{\varepsilon_n})_{n\in\mathbb{N}}$ with $\varepsilon_n := x_n$ converges to $\frac12(\max\{0,x\})^2$. But $u_{P_1}^{\varepsilon_n}(1,0) = 0$ implies $u^0(1,0) = 0$. Which contradicts $P_1 \in \mathcal{R}^+$.
(ii) Assume by contradiction that there exists $(P_n = (x_n,t_1))_{n\in\mathbb{N}}$ converging to $P_1 \in \mathcal{S}_0$ such that $u(P_n) = 0$. The blow-up sequence $(u_{P_1}^{\varepsilon_n})_{n\in\mathbb{N}}$ with $\varepsilon_n := x_n$ converges to $\frac12 x^2$. But $u_{P_1}^{\varepsilon_n}(1,0) = 0$ implies $u^0(1,0) = 0$. Which contradicts $P_1 \in \mathcal{S}_0$. □

Proof of Lemma 4.2. First recall that if u is non-increasing the free boundary only contains regular points and points of $\mathcal{S}_0$ (see Section 1). In $t=\varepsilon$ consider a point $P_i = (x_i, \varepsilon)$ of Γ.

If P_i belongs to $\mathcal{S}_0$: by Lemma 4.3 (ii), there exists r such that u is positive in $(x_i - r, x_i + r) \times [\varepsilon, T]$. In this case we pose $\tilde t = \varepsilon$, $x_i^- = x_+^i = x_i$. And there exists δ such that $\{u=0\} \cap ([x_i^-, x_i^+] \times [\varepsilon, T])$ is the straight line $x_i \times [\varepsilon, \varepsilon+\delta]$.

If P_i belongs to $\mathcal{R}$: by symmetry consider $P_i \in \mathcal{R}^-$. By Theorem 1.1, Γ is a $\mathcal{C}^{\frac12}$-uppergraph locally around P_i. We can extend this property at $C_\Gamma(P_i)$, the connected component of Γ which contains P_i. Denote $P^* = (x^*, t^*) := \inf_t\{P \in C_\Gamma(P_i) \,:\, P \notin \mathcal{R}^-\}$. If $t^* = T$, $C_\Gamma(P_i)$ is a $\mathcal{C}^{\frac12}$-uppergraph. If not: thanks to Lemma 3.5, P^* cannot belong to $\mathcal{R}^+$. So P^* belongs to $\mathcal{S}_0$. We set here $x_i = x_i^-$. The set $C_\Gamma(P_i)$ is a graph of class $\mathcal{C}^{\frac12}([0,t^*])$, let us denote it g_i^-.

Let us now deal with the other branch of the connected component of $\{u=0\}$. By Lemma 4.3, there is only a finite number of point of $\mathcal{R} \cup \mathcal{S}_0$ in $\{t=\varepsilon\}$. Define $x_i^+ := \sup_{x > x_i^-}\{(x,t) \,:\, (x,t) \in \{u=0\}\}$. By definition of x_i^+, $u=0$ in $[x_i^-, x_i^+] \times \{t=\varepsilon\}$. By Lemma 4.3 (i), (x_i^+, ε) is a point of $\mathcal{R}^+$. The previous argument gives the existence of a graph, g_i^+ of class $C^{\frac12}([0,t_2^*])$ with $t_2^* \in [\varepsilon, T]$.

If we cannot extend the $\mathcal{C}^{\frac{1}{2}}$ regularity of the graph up to $t = T$ it means that one at least of the two connected component of Γ defined above contains a point of $\mathcal{S}_0$. By symmetry we can assume that $t^* \leq t_2^*$. Assume by contradiction that the two curves do not meet in $(g_i^-(t^*), t^*)$. Then there exists $\bar{P} = (\bar{x}, t^*)$ in the connected component of Γ which contains (x_i^+, ε) such that $u \equiv 0$ in $[\tilde{x}, \bar{x}]$. But this is a contradiction with Lemma 4.3 (ii). $\square$

Proof of Theorem 4.1. In the generalised Black–Scholes model where σ and r do not depend on time, $u(x,t) = \Pi(e^x, T-t) - \phi(e^x)$ is non-decreasing in time on $[0,T]$ (see Proposition 5 in [23]) and is solution almost everywhere in $\mathbb{R} \times [0,T]$ of (1.4) with initial condition $u(\cdot, 0) = 0$. The Assumption (1.6) is satisfied with a, b, c and f given from σ, r and ϕ by (1.5). In [17], J. Harrison and S. Pliska proved the equivalence between the completeness of the market and $\sigma > 0$. Theorem 4.1 is so a direct consequence of Lemma 4.2. $\square$

5. Application to American vanilla options

The two most classical pay-off functions are the put (*i.e.* $\phi(S_t, t) = \max\{0, K - S_t\}$) and the call (*i.e.* $\phi(S_t, t) = \max\{0, S_t - K\}$) where the fixed price K is called *the strike*. Our results apply to these pay-off.

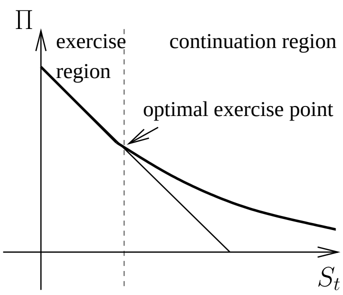
Theorem 5.1 (Exercise boundary for American vanilla options are regular) *In the generalised Black–Scholes model where σ and r do not depend on time. Assume $\sigma \in L^\infty(\mathbb{R} \times [0,T])$ is such that $x\frac{d\sigma}{dx}$ is bounded in $L^\infty(\mathbb{R} \times [0,T])$. Then*

(i) *If the underlying asset is solution of (1.1), the exercise boundary of an American put is a $\mathcal{C}^{\frac{1}{2}}$-subgraph in time for all $t < T$.*

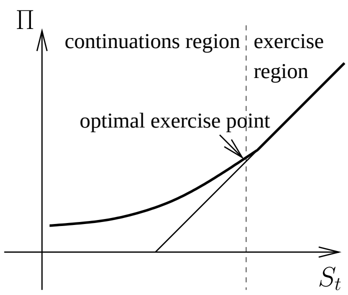
(ii) *If the underlying asset is solution of $dS_t = (r(S_t) - \delta)\, S_t\, dt + \sigma(S_t)\, S_t\, dW_t$, the exercise boundary of an American call is a $\mathcal{C}^{\frac{1}{2}}$-uppergraph in time for all $t < T$.*

Proof. The coefficients a, b and c defined from r and σ as in (1.5) are in $\mathcal{H}^\alpha(\mathbb{R} \times [0,T])$. The market is complete, so $a > 0$.

The price of the option Π is non-increasing in time. Due to the shape of the pay-off (see figure below) the exercise boundary is non-decreasing in time for the American put and non-increasing for the American call.



American put



American call

In particular the exercise boundary is a graph in time. For the American put, from (1.5) we compute $f(x,t)\mathbb{1}_{\{\Pi > \phi\}}(x,t) = rK\mathbb{1}_{\{\Pi > \phi\}}\mathbb{1}_{\{t > s(t)\}} = rK\mathbb{1}_{\{\Pi > \phi\}}$ because s in non-decreasing and $\lim_{t \to T} s(t) = K$. So we can consider the obstacle $\tilde{f} = rK$ in (1.4), $\tilde{f}$ is regular and non-degenerate. For the American call, from (1.5) we compute $f(x,t) = r\min\{K, \frac{rK}{\delta}\}\mathbb{1}_{\{\Pi > \phi\}}\mathbb{1}_{\{x > K\}} = r\min\{K, \frac{rK}{\delta}\}\mathbb{1}_{\{\Pi > \phi\}}$ for $t \leq T - \varepsilon$

because s in non-decreasing and $\lim_{t\to T} s(t) = \min\{K, \frac{rK}{\delta}\}$ (see [10]). So we can consider the obstacle $\tilde{f} = r\min\{K, \frac{rK}{\delta}\}$ in (1.4) for $t \leq T-\varepsilon$, $\tilde{f}$ is regular and non-degenerate.

For the American put: let $P_1 = (x_1, t_1) \in [0, T)$ be a point of the exercise boundary. There exists $r > 0$ such that $\Pi = \phi$ in $[x_1 - r, x_1] \times \{t = t_1\}$. But Π is non-increasing in time so there exists $\delta > 0$ such that $\Pi = \phi$ in $[x_1 - r, x_1] \times \{t_1, T\}$. So for r small enough

$$\frac{|\{\Pi > \phi\} \cap Q_r(P_1)|}{|Q_r(P_1)|} \supset \frac{|[x_1 - r, x_1] \times [t_1, t_1 + r^2]|}{|Q_r(P_1)|} = \frac{1}{4}$$

So the density of $\{\Pi > \phi\}$ in P_1 is non-zero and P_1 is not in $\mathcal{S}_0$. As u is non-increasing in time there is no point of $\mathcal{S} \setminus \mathcal{S}_0$ so the exercise boundary of the American put is only made of point of $\mathcal{R}^+$. Hence the exercise boundary is a $\mathcal{C}^{\frac{1}{2}}$-subgraph in time for all $t < T$.

Similarly for the American call: for r small enough $\{\Pi > \phi\} \cap Q_r(P_1) \supset [x_1, x_1 + r] \times [t_1, t_1 + r^2]$. So there is no point of $\mathcal{S}_0$. Hence the exercise boundary of the American call is made of point of $\mathcal{R}^-$, and is a $\mathcal{C}^{\frac{1}{2}}$-uppergraph in time for all $t < T$.

Acknowledgements. The author would like to thank J. Dolbeault and R. Monneau for their fruitful comments and suggestions. The author would also like to thank D. Lamberton and B. Lapeyre and theirs students for their enlightening discussion. This study has been partially supported by the ACI NIM "EDP et finance, # 2003-83".

References

[1] Y. Achdou, *An inverse problem for parabolic variational inequalities in the calibration of american options.* to appear in SIAM journal of Control and Optimization.

[2] G. Barles, J. Burdeau, M. Romano, and N. Samsœn, *Estimation de la frontière libre des options américaines au voisinage de l'échéance*, C. R. Acad. Sci. Paris Sér. I Math., 316 (1993), pp. 171–174.

[3] A. Bensoussan, *On the theory of option pricing*, Acta Appl. Math., 2 (1984), pp. 139–158.

[4] A. Bensoussan and J.-L. Lions, *Applications des inéquations variationnelles en contrôle stochastique*, Dunod, Paris, 1978. Méthodes Mathématiques de l'Informatique, No. 6.

[5] H. Berestycki, J. Busca, and I. Florent, *Asymptotics and calibration of local volatility models*, Quant. Finance, 2 (2002), pp. 61–69. Special issue on volatility modelling.

[6] T. Björk, *Interest rate theory*, in Financial mathematics (Bressanone, 1996), vol. 1656 of Lecture Notes in Math., Springer, Berlin, 1997, pp. 53–122.

[7] A. Blanchet, J. Dolbeault, and R. Monneau, *On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients.* Preprint HAL # 4421 (2005).

[8] L. Caffarelli, A. Petrosyan, and H. Shahgholian, *Regularity of a free boundary in parabolic potential theory*, J. Amer. Math. Soc., 17 (2004), pp. 827–869 (electronic).

[9] L. A. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math., 139 (1977), pp. 155–184.

[10] E. Chevalier, *Critical price near maturity for an american option on a dividend-paying stock in a local volatility model*, Mathematical Finance, 15 (2005), pp. 439–463.

[11] B. Dupire, *Pricing and hedging with smiles*, in Mathematics of derivative securities (Cambridge, 1995), vol. 15 of Publ. Newton Inst., Cambridge Univ. Press, Cambridge, 1997, pp. 103–111.

[12] N. El Karoui, *Les aspects probabilistes du contrôle stochastique*, in Ninth Saint Flour Probability Summer School—1979 (Saint Flour, 1979), vol. 876 of Lecture Notes in Math., Springer, Berlin, 1981, pp. 73–238.

[13] E. Fournié, J.-M. Lasry, J. Lebuchoux, and P.-L. Lions, *Applications of Malliavin calculus to Monte-Carlo methods in finance. II*, Finance Stoch., 5 (2001), pp. 201–236.

[14] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[15] ———, *Parabolic variational inequalities in one space dimension and smoothness of the free boundary*, J. Functional Analysis, 18 (1975), pp. 151–176.

[16] ———, *Variational principles and free-boundary problems*, Robert E. Krieger Publishing Co. Inc., Malabar, FL, second ed., 1988.

[17] J. M. Harrison and S. R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Process. Appl., 11 (1981), pp. 215–260.

[18] I. Karatzas, *On the pricing of American options*, Appl. Math. Optim., 17 (1988), pp. 37–60.

[19] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, R.I., 1967.

[20] D. Lamberton and S. Villeneuve, *Critical price near maturity for an American option on a dividend-paying stock*, Ann. Appl. Probab., 13 (2003), pp. 800–815.

[21] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[22] J. Mc Kean, H.P., *Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics*, Indust. Manage. Rev., 6 (1965), pp. 32–39.

[23] G. Rapuch, *American option and the free boundary exercise region : a pde approach*, Interfaces Free Bound., 7 (2005), pp. 79–98.

[24] P. Van Moerbeke, *An optimal stopping problem with linear reward*, Acta Math., 132 (1974), pp. 111–151.

[25] S. Villeneuve, *Options américaines dans un modèle de Black-Scholes multidimensionnel*, PhD thesis, Université De Marne la Vallée, 1999.

[26] G. S. Weiss, *A homogeneity improvement approach to the obstacle problem*, Invent. Math., 138 (1999), pp. 23–50.